$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{14} & a_{24} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{13} & a_{23} & a_{43} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}$$

ON A SPECIAL CASE OF EQUILIBRIUM OF A FLEXIBLE, INEXTENSIBLE STRING.

By CHARLES C. STECK, Durham, New Hampshire.

It is proposed in this paper to discuss the curve of equilibrium of a weightless, flexible, inextensible string, the end points of which are attached to two fixed points, each element of the string being acted upon by a force always normal to an axis which it meets, this force being any function of the distance r of the element from the given axis. A special discussion is made of the case where the force varies inversely as r^3.

The general problem as here stated is suggested by Appell in his *Traité de Mécanique Rationelle*, Vol. I, Chap. VII. In connection with this problem he gives the three following equations which are directly obtainable from the results he has arrived at in his discussion of equilibrium of strings:

$$(1)\quad T\frac{dz}{ds}=c; \quad (2)\quad Tr^2\frac{d\theta}{ds}=k; \quad (3)\quad T=-\int F(r)dr-h=\phi(r),$$

in which T is the tension at any point, $F(r)$ the force, the z-axis the axis to which the force is normal, r and θ the polar co-ordinates in the xy plane, and c, k, h arbitrary constants.

Starting with these three equations and with the equation

$$(4)\qquad ds^2=dr^2+r^2d\theta^2+dz^2,$$

we have, on eliminating T,

$$(5)\qquad k\frac{dr}{rd\theta}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}},$$

$$cr\frac{dr}{dz}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}, \tag{6}$$

$$r\phi(r)\frac{dr}{dz}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}. \tag{7}$$

On integrating these equations we obtain

$$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}}, \tag{8}$$

$$z-z_0=\pm c\int\frac{rdr}{\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}}, \tag{9}$$

$$s-s_0=\pm\int\frac{r\phi(r)dr}{\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}}}. \tag{10}$$

Equations (8) and (9) define the curve of equilibrium of the string. Equation (8) is the equation of a cylindrical surface the elements of which are perpendicular to the plane xy. Equation (9) is the equation of a surface of revolution, the axis of revolution being the z-axis. Equations (8), (9), and (10) contain six arbitrary constants which are to be determined from initial conditions.

Special Cases.

Case I. $c=0$, $k\neq 0$.

From (1) when $c=0$, either $T=0$ or $\frac{dz}{ds}=0$. Since T cannot be zero when $F(r)$ is not zero, $\frac{dz}{ds}$ must be zero; that is, the curve of equilibrium is in a plane perpendicular to the z-axis. Equations (8), (9), and (10) take the following forms:

$$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{[\phi(r)]^2r^2-k^2\}}}, \tag{11}$$

$$z-z_0=0, \tag{12}$$

$$s-s_0=\pm\int\frac{r\phi(r)dr}{\sqrt{\{[\phi(r)]^2r^2-k^2\}}}. \tag{13}$$

Equations (11) and (12) define the curve of equilibrium of a string under the action of a central force.

Case II. $k=0$, $c\neq 0$, $r\neq 0$.

From (2), when $k=0$, either $T=0$ or $r^2\frac{d\theta}{ds}=0$. As in Case I, T cannot be zero, and it follows that $r^2\frac{d\theta}{ds}$ must be zero; that is, $\frac{d\theta}{ds}=0$ or $\theta=\theta_0$. The curve of equilibrium lies in a plane perpendicular to the xy plane and passing through the z-axis. Equations (8), (9), and (10) take the forms

$$\theta-\theta_0=0, \tag{14}$$

$$z-z_0=\pm c\int\frac{dr}{\sqrt{\{[\phi(r)]^2-c^2\}}}, \tag{15}$$

$$s-s_0=\pm\int\frac{\phi(r)\,dr}{\sqrt{\{[\phi(r)]^2-c^2\}}}. \tag{16}$$

Case III. $k=0$, $c=0$, $r\neq 0$.

For this case equations (8), (9), and (10) take the forms,

$$\theta-\theta_0=0, \tag{17}$$

$$z-z_0=0, \tag{18}$$

$$s-s_0=r. \tag{19}$$

The curve of equilibrium is a straight line perpendicular to the z-axis. Since this is true whatever be the force, so long as it is a function of r and r is nowhere zero, it will be unnecessary to consider this case in the discussion which follows and which deals with a special force.

Discussion of the Case Where $F(r)=\frac{\mu}{r^2}$, $\mu>0$.

Since we shall deal only with finite forces, it follows that r can nowhere be zero; that is, the string cannot touch the z-axis at any point. Further, since $\mu>0$, we shall have a repulsive force. This constant μ is the value of the force at a unit's distance from the z-axis. Substituting the above value of $F(r)$ in equation (3) and integrating, we have

$$\phi(r)=\frac{\mu}{r}-h=T. \tag{20}$$

Substituting this value of $\phi(r)$ in equations (8), (9), and (10), we get,

$$\theta-\theta_0=\pm k\int\frac{dr}{r\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}, \tag{21}$$

$$z-z_0=\pm c\int\frac{rdr}{\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}, \tag{22}$$

$$s-s_0=\pm\int\frac{(\mu-hr)dr}{\sqrt{\{r^2(h^2-c^2)-2h\mu r+\mu^2-k^2\}}}. \tag{23}$$

These three indicated integrations can be immediately performed. The following cases must be distinguished.

I. $h^2>c^2$, $\mu^2>k^2$.
   (a) $c^2\neq0$, $k^2\neq0$.
   (b) $c^2=0$, $k^2\neq0$.
   (c) $c^2\neq0$, $k^2=0$.

II. $h^2>c^2$, $\mu^2=k^2$.
   (a) $c^2\neq0$.
   (b) $c^2=0$.

III. $h^2>c^2$, $\mu^2<k^2$.
   (a) $c^2\neq0$.
   (b) $c^2=0$.

IV. $h^2<c^2$, $\mu^2>k^2$.
   (a) $h^2\neq0$, $k^2\neq0$.
   (b) $h^2=0$, $k^2\neq0$.
   (c) $h^2\neq0$, $k^2=0$.
   (d) $h^2=0$, $k^2=0$.

V. $h^2<c^2$, $\mu^2=k^2$.
   (a) $h^2\neq0$.
   (b) $h^2=0$.

VI. $h^2<c^2$, $\mu^2<k^2$.
   (a) $h^2\neq0$.
   (b) $h^2=0$.

VII. $h^2=c^2$, $\mu^2>k^2$.
   (a) $h^2\neq0$, $k^2\neq0$.
   (b) $h^2=0$, $k^2\neq0$.
   (c) $h^2\neq0$, $k^2=0$.

VIII. $h^2=c^2$, $\mu^2=k^2$.
   (a) $h^2\neq0$.
   (b) $h^2=0$.

IX. $h^2=c^2$, $\mu^2<k^2$.
   (a) $h^2\neq0$.
   (b) $h^2=0$.

The following notations are used:

$$R\equiv r^2(h^2-c^2)-2h\mu r+\mu^2-k^2,$$
$$m^2\equiv h^2-c^2 \text{ if } h^2>c^2,$$
$$-m^2\equiv h^2-c^2 \text{ if } h^2<c^2,$$
$$n^2\equiv\mu^2-k^2 \text{ if } \mu^2>k^2,$$
$$-n^2\equiv\mu^2-k^2 \text{ if } \mu^2<k^2,$$
$$\Delta\equiv h^2\mu^2-(h^2-c^2)(\mu^2-k^2).$$

We give here an outline of the method used in determining the constants for Case I (a). The results for this case and for all of the others are put in tabular form at the end, followed by a series of notes by which necessary explanations are made.

DETERMINATION OF THE CONSTANTS FOR CASE I (a).

Initial conditions. Let the initial value of r be that value which makes $R=0$. Call this value r_1. When $r=r_1$, let $\theta=0$, $z=0$, $s=0$, $T=T_1$.

Interpretation of initial conditions. Returning to equations (1) and (2), we see that when $R=0$, since $k\neq 0$ and $c\neq 0$, $\frac{1}{r}\frac{dr}{d\theta}=0$ and $\frac{dr}{dz}=0$. Now $\frac{1}{r}\frac{dr}{d\theta}$ is the cotangent of the angle between the radius vector and the tangent to the guiding curve of the cylinder at any point. We shall call this angle ϕ. $\frac{dr}{dz}$ gives the slope of the tangent to the meridian section of the surface of revolution at the same point. Call this angle ψ. With the initial conditions just imposed $\phi_1=\frac{1}{2}\pi$ and $\psi_1=0$.

From (1) it is evident that c is the value of the tension when $\frac{dz}{ds}=1$; *i. e.*, the component of the tension parallel to the z-axis; call it T_0. c may always be taken positively if we measure s in such a direction that z and s increase simultaneously since T is essentially positive. The constant h may be found from equation (20). We have, on imposing initial conditions

$$h=\frac{\mu}{r_1}-T_1.$$

The constant k is found from the equation $R=0$. Solving this equation for k^2 we get

$$k^2=r_1^2(T_1^2-T_0^2).$$

The values of θ_0, z_0, and s_0 are

$$\theta_0=0,\ z_0=0,\ s_0=0.$$

The corresponding values of m^2, n^2, and J are

$$m^2=h^2-c^2=\frac{1}{r_1^2}[\mu^2-2\mu r_1T_1+r_1^2(T_1^2-T_0^2)],$$

$$n^2=\mu^2-k^2=\mu^2-r_1^2(T_1^2-T_0^2),$$

$$J=[\mu T_1-r_1(T_1^2-T_0^2)]^2.$$

Case		r_1	Equation of Cylinder	θ_0	Equation of Surface of Revol.	Z_0	Equation of Length of Arc	S_0	Notes
I $h^2>c^2+\frac{k^2}{4}h^2$	a $c^2\neq 0$ $k^2\neq 0$	$\frac{h\mu+\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh^2\frac{z}{k}\frac{m}{n}(\theta-\theta_0)}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\frac{h\mu-m^2r}{\sqrt{\Delta}}\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\cosh^{-1}\frac{h\mu-m^2r}{\sqrt{\Delta}}-\frac{h\sqrt{R}}{m^3}\right\}$	c	A B
	b $c^2=0$ $k^2\neq 0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh^2\frac{z}{k}\frac{m}{n}(\theta-\theta_0)}$	0	$z-z_0=0$	0	$s-s_0=\mp\frac{\sqrt{R}}{h}$	0	A B
	C $c^2\neq 0$ $k^2=0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$\theta-\theta_0=0$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\frac{h\mu-m^2r}{\sqrt{\Delta}}\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\frac{h\mu-m^2r}{\sqrt{\Delta}}-\frac{h\sqrt{R}}{m^3}\right\}$	0	A B
II $h^2>c^2$ $k^2\neq 0$	a $c^2\neq 0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{2\mu h}{m^2-h^2(\theta-\theta_0)^2}$	$\pm\frac{c}{h}\tan\psi_1$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\frac{h\mu-m^2r}{h\mu}\right\}$	$\mp\frac{c}{m^2}\left\{\frac{k\cot\phi_1}{h}\cosh^{-1}\frac{h\mu-m^2r}{h\mu}+\frac{h\sqrt{R}}{m^2}\right\}$	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\frac{h\mu-m^2r}{h\mu}-\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\left[\frac{k\mu}{m^2}\cosh^{-1}\frac{h\mu-m^2r}{h\mu}-\frac{h\sqrt{R}}{m^2}\right]$	C D E
	b $c^2=0$	$\frac{k\mu}{h^2}\tan\phi_1$	$r=\frac{2\mu}{h[1-(\theta-\theta_0)^2]}$	$\pm\frac{c}{h}\tan\psi_1$	$z-z_0=0$	0	$s-s_0=\mp\frac{\sqrt{R}}{h}$	$s_0=\mp\frac{k}{h}c\cot\phi_1$	C D
III $h^2>c^2$ $k^2=0$	a $c^2\neq 0$	$\frac{h\mu+\sqrt{\Delta}}{m^2}$	$r=-\frac{n^2}{h\mu-\sqrt{\Delta}\cos^2\frac{z}{k}\theta}$	$\frac{k}{h}\cdot\frac{\pi}{2}$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\cosh^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)\right\}$	0	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1-\frac{h^2}{m^2}\right)\cosh^{-1}\frac{m^2r-h\mu}{\sqrt{\Delta}}-\frac{h\sqrt{R}}{m^3}\right\}$	0	C D F
	b $c^2=0$	$\frac{h\mu+\sqrt{\Delta}}{h^2}$	$r=\frac{-n^2}{h\mu-\sqrt{\Delta}\cos^2\frac{z}{k}\theta}$	$\frac{k}{n}\cdot\frac{\pi}{2}$	$z-z_0=0$	0	$s-s_0=\mp h\frac{\sqrt{R}}{m^2}$	0	C D F
IV $h^2<c^2$ $\mu^2>h^2$	a $k^2\neq 0$ $k^2\neq 0$	$\frac{h\mu+\sqrt{\Delta}}{m^2}$	$r=\frac{n^2}{h\mu+\sqrt{\Delta}\cosh\left[\frac{2}{k}\frac{m}{n}(\theta-\theta_0)\right]}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{c}{m^3}h\mu\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	A
	b $k^2=0$ $k\neq 0$	$\frac{n}{m}$	$r=\frac{n}{m\cosh\left[\frac{2}{k}\frac{m}{n}(\theta-\theta_0)\right]}$	0	$r^2+(z-z_0)^2=r_1^2$	0	$s-s_0=\pm\left\{\frac{\mu}{m}\sin^{-1}\frac{mr}{n}\right\}$	0	A
	C $k^2\neq 0$ $k^2=0$	$\frac{h\mu+\sqrt{\Delta}}{m^2}$	$\theta-\theta_0=0$	0	$z-z_1=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{c}{m^3}h\mu\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	A
	d $k^2=0$ $k^2=0$	$\frac{\mu}{c}$	$\theta-\theta_0=0$	0	$\frac{r^2}{\mu^2}+\frac{(z-z_0)^2}{c^2\mu^2}=1$	0	$s-s_0=\pm\left(\frac{\mu}{c}\sin^{-1}\frac{c}{\mu}r\right)$	0	A
V $h^2<c^2$ $\mu^2=h^2$	a $k^2\neq 0$	$\frac{-2h\mu}{m^2}$	$r=\frac{-2h\mu}{h^2(\theta-\theta_0)^2+m^2}$	0	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{c}{m^3}h\mu\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r+h\mu}{\sqrt{\Delta}}\right)-\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	C
	b $k^2=0$								G
VI $h^2<c^2$ $\mu^2<h^2$	a $k^2\neq 0$	$\frac{h\mu-\sqrt{\Delta}}{m^2}$	$r=-\frac{n^2}{h\mu-\sqrt{\Delta}\cos^2\frac{n}{k}\theta}$	$\frac{k}{n}\cdot\frac{\pi}{2}$	$z-z_0=\pm c\left\{\frac{\sqrt{R}}{m^2}+\frac{h\mu}{m^3}\sin^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)\right\}$	$\mp\frac{c}{m^3}h\mu\cdot\frac{\pi}{2}$	$s-s_0=\pm\left\{\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\sin^{-1}\left(\frac{m^2r-h\mu}{\sqrt{\Delta}}\right)+\frac{h\sqrt{R}}{m^2}\right\}$	$\mp\frac{\mu}{m^2}\left(1+\frac{h^2}{m^2}\right)\frac{\pi}{2}$	C F
	b $k^2=0$								G
VII $h^2=c^2$ $\mu^2\neq h^2$	a $k^2\neq 0$ $k^2\neq 0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{-n^2}{h\mu}\cdot\frac{1}{\cos^2\frac{1}{k}\frac{n}{2}(\theta-\theta_0)}$	$\pm\frac{k}{n}\frac{\cot\phi_1+\sin\psi_1}{\cos\psi_1}$	$(z-z_0)^2=\frac{1}{9h^2\mu^2}(n^4+3h^2\mu^2r^2-2h\mu^2r^3)$	$\mp\frac{c}{3\mu^2}(n^2+h\mu r_1)r_1$	$s-s_0=\pm\frac{n^3+h\mu r-3\mu^2\sqrt{R}}{3h\mu^2}$	$\mp\frac{2\mu^2-h^2}{3\mu^2}h\mu r_1$	C E
	b $k^2=0$ $k^2\neq 0$	Z_1	$r=e^{\frac{2}{k}(\theta-\theta_0)}$	0	$z-z_0=0$	0	$s=s_0=\pm\frac{(\mu r-hr^2)}{2}$	$\mp\left(\frac{\mu-h}{2}\right)$	C
	c $k^2\neq 0$ $k^2=0$	$\frac{\mu}{h}$	$\theta-\theta_0=0$	0	$(z-z_0)^2=\frac{1}{9h^2}(\mu^2-3h^2\mu r^2+2h^3r^3)$	0	$s-s_0=\frac{\pm n^3+h\mu r-3\mu^2\sqrt{R}}{3h\mu^2}$	$\mp\frac{(2\mu^2-h\mu r)}{3\mu^2}r_1$	C
$VIII$ $h^2=c^2$ $\mu^2=h^2$	a $k^2\neq 0$	$\frac{k\cot\phi_1}{c\tan\psi_1}$	$r=\frac{-2k^2}{\mu h(\theta-\theta_0)^2}$	$\frac{-2\mu}{\sqrt{-2\mu h}}$	$(z-z_0)^2=\frac{2}{9}\frac{c}{\mu}r^3$	$\mp\frac{2}{3}\frac{r_1^2h}{\sqrt{-2\mu h}}$	$(s-s_0)^2=\frac{-2r}{9h\mu}(3\mu-hr)^2$	$\mp\frac{2r^2}{3\sqrt{-2\mu h}}(3\mu-hr)$	C
	b $k^2=0$	r_1	$r=r_1$		$z-z_0=0$	0			C H
IX $h^2=c^2$ $\mu^2<h^2$	a $k^2\neq 0$	$\frac{-n^2}{2h\mu}$	$r=\frac{-n^2}{\mu h}\cdot\frac{1}{1+\cos\frac{2}{k}(\theta-\theta_0)}$	0	$(z-z_0)^2=\frac{1}{9\mu^2h^2}(2h^3\mu^2r^3+3h^2\mu^2r^2)$	0	$s-s_0=\pm\frac{h\mu r-n^2-3\mu^2\sqrt{R}}{3h\mu^2}$	$\mp\frac{(2\mu^2-h^2)h\mu r_1}{3\mu^2}$	C F
	b $k^2=0$								G

NOTES ON PRECEDING TABLE.

A. $h>0$.

B. $m<0$; $n>0$.

C. $h<0$.

D. $m>0$; $n<0$.

E. Since the value of r_1 used in Case I has become zero in this case, and since this contradicts the hypothesis that $F(r)$ is finite, it follows that $\phi_1 \neq \frac{1}{2}\pi$, $\psi_1 \neq 0$. The initial value of r cannot be a root of the equation $R=0$. When $r=r_1$, let $\phi=\phi_1 \neq \frac{1}{2}\pi$, $\psi=\psi_1 \neq 0$. With other initial conditions as before, we obtain for k the value

$$k=r_1 T_0 \frac{\tan \phi_1}{\cot \phi_1}.$$

F. In determining the equation of the cylinder for this case we have nsed the positive sign of equation (8) only. The negative sign gives a similar curve.

G. For $h=0$ the expression $\sqrt{R}$ becomes imaginary for all values of r.

H. To determine the curve of equilibrium for this case we have from equation (5), since $R=0$,

$$\frac{k}{r}\frac{dr}{d\theta}=0.$$

Since $k \neq 0$ and $r \neq 0$, we obtain on integration, $r=\text{constant}=r_1$.

Graphs of the guiding curves of the cylinder and a meridian section of the surface of revolution have been made for each case. Since there are so many cases to consider a reproduction of these graphs is not convenient here. In regard to the curve of equilibrium, the prevailing type is a spiral with the z-axis as the line about which it winds.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

357. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the system

$$\sqrt{(x^2+a^2+b^2+c^2)}=\sqrt{(y^2+b^2+c^2)}+\sqrt{(z^2+b^2+c^2)},$$
$$\sqrt{(y^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+c^2)}+\sqrt{(z^2+a^2+c^2)},$$
$$\sqrt{(z^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+b^2)}+\sqrt{(y^2+a^2+b^2)}.$$

Solution by B. F. FINKEL, Ph. D., Drury College.

Transposing the first term of the second member of the first equation, squaring, collecting, and transposing, we have

$$x^2+y^2-z^2+s^2=2\sqrt{(x^2+s^2)}\sqrt{(y^2+s^2-a^2)}\ldots(1),$$

where $s^2=a^2+b^2+c^2$. Similarly, we obtain from the second equation,

$$x^2+y^2-z^2+s^2=2\sqrt{(y^2+s^2)}\sqrt{(x^2+s^2-b^2)}\ldots(2),$$

and from the third,

$$x^2+z^2-y^2+s^2=2\sqrt{(z^2+s^2)}\sqrt{(x^2+s^2-c^2)}\ldots(3).$$

By transposing the second term of the first equation, squaring, collecting, and retransposing, we get

$$x^2+z^2-y^2+s^2=2\sqrt{(x^2+s^2)}\sqrt{(z^2+s^2-a^2)}\ldots(4).$$

From (1) and (2) we have,

$$\sqrt{(x^2+s^2)}\sqrt{(y^2+s^2-a^2)}=\sqrt{(y^2+s^2}\sqrt{(x^2+s^2-b^2)};$$

whence, by squaring and dividing by $(x^2+s^2)(y^2+s^2)$, we get

$$\frac{y^2+s^2-a^2}{y^2+s^2}=\frac{x^2+s^2-b^2}{x^2+s^2}.$$

Hence, $\frac{a^2}{y^2+s^2}=\frac{b^2}{x^2+s^2}$, or $y^2+s^2=\frac{a^2}{b^2}(x^2+s^2)$.

From (3) and (4), we obtain, in like manner,

$$z^2+s^2=\frac{a^2}{c^2}(x^2+s^2).$$

Substituting these values of z^2+s^2 and y^2+s^2 in the first equation, we have

$$\sqrt{(x^2+s^2)}=\sqrt{\frac{a^2}{b^2}(x^2+s^2)-a^2}+\sqrt{\frac{a^2}{c^2}(x^2+s^2)-a^2};$$

whence, $bc\sqrt{(x^2+s^2)}=ab\sqrt{(x^2+s^2-b^2)}+ab\sqrt{(x^2+s^2-c^2)}$.

Squaring,

$$b^2c^2(x^2+s^2)=a^2c^2(x^2+s^2-b^2)+2a^2bc\sqrt{(x^2+s^2-b^2)}\sqrt{(x^2+s^2-c^2)}+a^2b^2(x^2+s^2-c^2);$$

transposing, and combining,

$$(b^2c^2-a^2c^2-a^2b^2)(x^2+s^2)+2a^2b^2c^2=2a^2b^2c^2\sqrt{(x^2+s^2-b^2)}\sqrt{(x^2+s^2-c^2)}.$$

Squaring both members of this equation, and rearranging terms, we have

$$[(b^2c^2-a^2c^2-a^2b^2)^2-4a^4b^2c^2](x^2+s^2)+4a^2b^4c^4(x^2+s^2)=0.$$

$\therefore x^2+s^2=0$, or $x^2+s^2=$

$$\frac{4a^2b^4c^4}{4a^4b^2c^2-(b^2c^2-a^2c^2-a^2b^2)^2}$$

$$=\frac{4a^2b^4c^4}{(b^2c^2-a^2c^2-a^2b^2+2a^2bc)(-b^2c^2+a^2c^2+a^2b^2+2a^2bc)}$$

$$=\frac{4a^2b^4c^4}{[b^2c^2-a^2(b-c)^2][a^2(b+c)^2-b^2c^2)]}$$

$$= \frac{4a^2b^4c^4}{(bc+ab-ac)(bc-ab+ac)(ab+ac+bc)(ab+ac-bc)}$$

$= \frac{4a^2b^4c^4}{\Delta}$, where Δ is the denominator of the above fraction.

$$y^2+s^2 = \frac{4a^4b^2c^4}{\Delta}, \text{ and } z^2+s^2 = \frac{4a^4b^4c^2}{\Delta}.$$

$$\therefore x = \pm\left(\frac{4a^4b^2c^4-s^2\Delta}{\Delta}\right)^{\frac{1}{2}}, \quad y = \pm\left(\frac{4a^4b^2c^4-s^2\Delta}{\Delta}\right)^{\frac{1}{2}}, \text{ and}$$

$$z = \pm\left(\frac{4a^4b^4c^2-s^2\Delta}{\Delta}\right)^{\frac{1}{2}}.$$

$x^2+s^2=0$ is not admissible.

PROBLEMS FOR SOLUTION.

ALGEBRA.

363. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

(a) If a and n be positive integers, the integral part of $[a+\sqrt{(a^2-1)}]^n$ is odd.

(b) If a and n be positive integers, the integral part of $[\sqrt{(a^2+1)}+a]^n$ is odd when n is even and even when n is odd. [From Todhunter's *Algebra*, p. 353].

364. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The English physicist, Hooke, published the discovery contained in the Latin sentence, "Ut tensio sic vis" by the cypher *cciiinosssttuv*. Preserving the lexicographical order, find which permutation, taking all letters, the Latin sentence is from the cypher.

365. Proposed by C. N. SCHMALL, New York City.

In still water, a steam tug goes 6 miles an hour less when towing a barge than when alone. Having drawn the barge 30 miles up a stream, whose current runs 1 mile an hour, it returns alone and completes the journey in 12 8/11 hours. Find the rate of the tug in still water.

GEOMETRY.

396. Proposed by DANIEL KRETH, Oxford, Iowa.

In the triangle ABC, $AB=214$, $BC=263$, and $AC=405$. A point P is situated in the same horizontal plane; angle $BPA=13^\circ\ 30'$ and angle $BPC=29^\circ\ 50'$. Find the distances, AP, BP, and CP.

397. Proposed by DAVID F. KELLEY, New York City.

If ABC be a semicircle and CD a perpendicular from C on the diameter AB, prove that the radius of the circle inscribed in the triangle ABC equals half the sum of the radius of the circle touching arc AC and the sides AD and DC of the triangle ADC, and the radius of the circle touching arc CB and sides DB and DC of triangle CDB, and that the centers of the three circles are collinear.

398. Proposed by C. N. SCHMALL, New York City.

In a square $ABCD$ draw the diagonal AC. Now bisect AD in G and draw GB cutting AC in H. Prove that $\triangle ACH=\frac{1}{2}\triangle CGH=\frac{1}{3}\triangle ABG=\frac{1}{4}\triangle BCH$.

CALCULUS.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

319. Proposed by C. N. SCHMALL, New York City.

Given $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$, prove

$$\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{vmatrix} = 4xyz.$$

MECHANICS.

265. Proposed by A. H. HOLMES, Brunswick, Maine.

A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Show that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

266. Proposed by A. M. HARDING, Assistant Professor of Mathematics, University of Arkansas.

A, B, C are three equidistant smooth pegs in the same horizontal line, and a heavy uniform string has its ends tied to A, C, and is looped over B. Show that there may or may not be a position of equilibrium in which the two catenaries, AB, BC, are unequal, and if there is such a position it will be stable. Show that the position of equilibrium in which the middle point of the string is at B is unstable or stable according as an unsymmetrical position of equilibrium does or does not exist. [Jeans' *Mechanics*, page 187].

AVERAGE AND PROBABILITY.

208. Proposed by A. M. HARDING, Assistant Professor of Mathematics, University of Arkansas.

Find the chance that the distance of two points within a square shall not exceed a side of the square.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

185. Proposed by R. D. CARMICHAEL, Bloomington, Ind.

Obtain the complete solution of the equatian $\phi(p^{\alpha}) = \phi(q^{\beta})$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes and α and β are unknown integers.

NOTES AND NEWS.

There has recently appeared from the publishers, Gauthier-Villars, a fine translation, by Paul Babarin, of Dr. Halsted's *Rational Geometry*. The book since its publication in English a few years ago, has been translated in German and Japanese, and now in French. In the preface of the French edition, C. A. Laisant says, among other things, that the book is clearly written, and the translation faithful and limpid, and he expresses the belief that the work is destined to exercise a profound effect upon the transformation of geometric teaching. F.

The reports of the American Committee of the International Commission on the Teaching of Mathematics, which are now being issued by the United States Bureau of Education, may be had upon application to the Commissioner of Education, Washington, D. C. Those who desire to receive these reports may thus not only get the half dozen already issued but may have their names entered so as to receive the remaining reports as fast as they are issued.

The final report of the International Commissioners is to be presented at the meeting of the International Congress of Mathematicians to be held in Cambridge, England, in August, 1912. S.

The annual resumé of doctorates conferred by American Universities, as printed in *Science* in the issue of August 18, 1911, should be of interest to readers of the MONTHLY.

It contains comparative tables showing the average number of doctorates in all subjects for each of 44 universities for the ten years, 1898-1907; the total number for the fourteen years, 1898-1911; the actual number for each of the years, 1908, 1909, 1910, 1911, together with the corresponding data for doctorates in the sciences including mathematics.

The universities which have conferred more than 100 doctorates during the last fourteen years are: Columbia 555, Chicago 545, Harvard 495, Yale 452, Johns Hopkins 411, Pennsylvania 341, Cornell 306, Wisconsin 152, Clark 137, and New York 123. Those showing an average of more than 10 during the ten years 1898-1907 are: Chicago 35.6, Harvard 33.8, Columbia 32.2, Yale 31.8, Johns Hopkins 30,5, Pennsylvania 22.5, and Cornell 18.1.

Those conferring more than 20 doctorates in 1911 are: Columbia 75, Chicago 55, Harvard 42, Cornell 34, Yale 31, Pennsylvania 29, Johns Hopkins 28.

In the sciences the institutions conferring a total of more than 100 doctorates in the last fourteen years are: Chicago 280, Johns Hopkins 239, Columbia 218, Harvard 198, Cornell 197, Yale 194, Pennsylvania 143, and Clark 126. Those having an average of more than 10 during the ten years, 1898-1907, are: Johns Hopkins 16.8, Chicago, 16.4, Harvard 14.1, Columbia 13.4, Yale 12.4, and Cornell 10.4. Those conferring 10 or more doctorates in the sciences in 1911 are: Chicago 35, Columbia 29, Cornell 27, Harvard 20, Johns Hopkins 19, Clark 16, Yale 15, Wisconsin 13, and Pennsylvania 10.

The total number of doctorates conferred in the last fourteen years by the 44 universities given in the tables is 4286, the total number in the sciences being 2037, or 48 per cent of the whole number. The total number for 1911 was 437, of which 239 were in the sciences, and the total average for the ten years, 1898-1907, was 272.4 of which 124.1 belong to the sciences.

The numbers in the various departments are also of interest. Those departments credited with a total of over 100 during the last fourteen years are chemistry 533, physics 264, zoölogy 244, psychology 222, mathematics 206, botany 183, English 121, geology 114, and history 105. Those departments showing more than 10 doctorates in 1911 are: chemistry 65, physics 37, English 33, history 26, philosophy 26, zoölogy 25, mathematics 25, psychology 23, education 23, botany 20, sociology 18, economics 16, geology 15, romance languages 12, Latin 11, and agriculture 11.

The following quotation from the report is significant of the trend with respect to the sciences:

"The number of doctorates in the natural and exact sciences is increasing more rapidly than in other subjects. Prior to 1908 the average number of degrees conferred in the sciences was 124, as compared with 198 in the other group, in the three following years the average numbers were 186 and 189, respectively; and this year the numbers were 239 and 198. As shown in the table, Chicago is the university which has conferred the largest number of degrees in the natural and exact sciences, followed by Johns Hopkins and Columbia. Of the degrees conferred by Cornell, 64 per cent have been in the sciences, at the Johns Hopkins 58 per cent, at Harvard 40 per cent, at Columbia 37 per cent. It is somewhat curious that the percentage at Wisconsin, Michigan, Illinois, and Minnesota should be as small as 42, 39, 54, and 37, respectively, as it is the general impression that the sciences are especially emphasized at the State universities."

The 25 doctorates in mathematics in 1911 were distributed as follows: Chicago 5, Yale 5, Clark 2, Johns Hopkins 2, Princeton 2, Pennsylvania 2, Columbia 1, Illinois 1, Michigan 1, Syracuse 1, Harvard 1, Cornell 1, and Cincinnati 1. S.

This issue of the MONTHLY was mailed January 13.

BOOKS AND PERIODICALS.

An Introduction to Thermodynamics. By John Mills, Professor of Physics and Electrical Engineering, Colorado College, Colorado Springs. 8vo. Cloth, viii+136 pages. Price, $2.00. Boston: Ginn & Co.

"This book is a clearly and carefully written presentation of the portions of elementary thermodynamics which are essential to the present-day engineer. It is suitable for use in third-year engineering courses or in regular college courses in thermodynamics. The text constitutes at the same time a complete course for general scientific or engineering students and an adequate preparation for further work in pure thermodynamics or in the special study of the design and operation of steam engines, turbines, heat engines, or compressed-air machines.

Although it makes use of elementary calculus it does not necessarily require a knowledge of calculus further than the fundamental concepts of a derivative and of an integration. It is intended to follow a good college course in general physics, but it includes a brief resume of such portions of this subject as are essential to the development of the subject of thermodynamics.

The text is illustrated by a careful selection of diagrams and plots. Illustrative problems cover the entire range of the material presented in the text and are accompanied by numerical solutions. These examples and the additional problems for solution are peculiarly strong and attractive features of the book."

Current Literature. A Review of the Times. Edited by Edward J. Wheeler. Price, $3.00 per year in advance.

Some of the important articles of the December number are, Bombarding our Judiciary; Socialist Victories at the Polls; and Big Business in a Big Quandary. F.

The American Review of Reviews. An International Monthly Magazine. Edited by Albert Shaw. Price, $3.00 per year in advance.

In addition to a resume of the affairs of the world, the December number contains articles on China in Revolution, Woman Suffrage Throughout the World, and Pacific Harbors and Panama Traffic. F.

McClure's Magazine. Edited by S. S. McClure. Price, $1.50 per year.

Among the important articles in the December number are, The Conflagration Hazard in New York, and A New Conscience and an Ancient Evil. F.

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VOL. XVIII. DECEMBER, 1911. NO. 12.

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JAN 19 1912

THE AMERICAN MATHEMATICAL MONTHLY.

PUBLISHED UNDER THE JOINT AUSPICES OF THE UNIVERSITY OF CHICAGO AND THE UNIVERSITY OF ILLINOIS.

EDITED BY
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OFFICE OF PUBLICATION: DRURY COLLEGE,
SPRINGFIELD, MISSOURI, U. S. A.

TERMS, $2.00 PER YEAR IN ADVANCE.
SINGLE COPIES, 25 CENTS.

A. HERMANN, Agent in Europe, Librarie de S. M. le Roi de Suede et de Norwege, 8 Rue de la Sorbonne, Paris.
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A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

Conducted with the Co-operation of the

AMERICAN PHYSICAL SOCIETY

By Edward L. Nichols, Ernest Merritt, and Frederick Bedell.

Edited and Published at Ithaca, New York.

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